Detailed description of the closing condition for the CMRA and the process for establishing the need for price increases

The closing condition

The CMRA bidding process ends when at least one value-maximising feasible combination is inclusive.

Let the vector $\mathbf{s} = (s_1 \dots s_L)$ denote the available supply of lots in categories $l = 1 \dots L$ in the relevant auction phase, the vector $\mathbf{r} = (r_1 \dots r_L)$ denote the respective reserve prices and the vector $\mathbf{p} = (p_1 \dots p_L)$ the respective lot prices in the most recent round.

Assume that there are $n = 1 \dots N$ bidders taking part in the process and let K^n denoted the number of bids submitted by bidder n up to this point, including both its headline bid and all additional bids.

Let $\beta_j^k = (P_j^k; \mathbf{q})$ be bid $k = 1 \dots K_j$ from bidder $j = 1 \dots N$ where P_j^k is the bid amount and $\mathbf{q}_j^k$ is a vector representing the bid package (i.e. the number of lots in in each category for which the bidder is prepared to pay the bid amount).

Let $B = \left\{\beta_j^k\right\} \forall\, j = 1 \dots N, k = 1 \dots K^j$ be the set of all bids received in the auction phase so far.

Let q_l denote the number of lots in category $l = 1 \dots L$ included in the bid package.

Note that the set of bids for a bidder may include a zero bid with $P = 0$ and $q_l = 0 \forall\, l = 1 \dots L$.

A combination of bids is defined by $\mathbf{x} = (x_1^1, x_1^2, \dots x_1^{K_1}, x_2^1, \dots, x_N^{K_N})$, where x_j^k is a binary variable that is set equal to 1 if the k-th bid from bidder j is included in the combination and zero otherwise.

In order to check whether the closing condition is met, we define the following:

A *feasible combination* is a combination of bids that satisfies the following conditions:

- Condition (1) - no excess demand: $\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k \mathbf{q}_j^k \leq \mathbf{s}$
- Condition (2) - no more than one bid per bidder: $\sum_{k=1}^{K_j} x_j^k \leq 1 \forall\, j = 1 \dots N$

The *value of a feasible combination* is then given by:

$$V(\mathbf{x}) = \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k P_j^k + \left(\mathbf{s} - \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k \mathbf{q}_j^k\right) \mathbf{r}.$$

Let X be the set of all feasible combinations, i.e. the set of all $\mathbf{x}$ that meet conditions (1) and (2).

$V^* = \max\{V(\mathbf{x}) : \mathbf{x} \in X\}$ is the maximum value of all feasible combinations and the *set of value-maximising feasible combinations* X^* is then given by $X^* = \{\mathbf{x}^* \in X : V(\mathbf{x}^*) = V^*\}$. Note that there will always be at least one value-maximising feasible combination.

The *set of inclusive combinations* X^I is defined as $X^I = \left\{\mathbf{x}^I \in X : \sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k = N\right\}$. i.e. the set of all feasible combinations that includes exactly one bid from each bidder. Note that X^I may be empty.

The closing condition is then defined as follows:

The bidding process ends if $|X^* \cap X^I| > 0$, i.e. there is at least one value-maximising feasible combination that is also inclusive.

Identifying lot categories for which prices need to increase

If the closing condition fails to hold, this implies that none of the value-maximising feasible combinations is inclusive. Therefore, in each of the value-maximising feasible combination there is at least one bidder for whom no bid is included in the combination. However, this may be because such a bid is tied with leaving the lots requested by the bidder unsold. Such combinations should be exccluded as we are interested only in those value-maximising feasible combinations in which it would not be possible to include bids from one or more of these bidders without discarding any bids from other bidders included in the combination (rather than simply by reducing the number of unsold lots).

Say that $\mathbf{x}$ dominates $\hat{\mathbf{x}}$ if $x_j^k \geq \hat{x}_j^k\ \forall\, j, k$ and let $X^{**} \subseteq X^*$ be the set of all value maximising feasible combinations that are not dominated.

We define *ommitted bidders* as bidders who are excluded from at least one combination in X^{**}.

Thus, the set of ommitted bidders, θ, is:

$$\theta = \left\{j = 1 \ldots N :\ \exists\ \mathbf{x} \in X^{**} :\ \sum_{k=1}^{K_j} x_j^k = 0\right\}$$

Let β_j^h denote the headline bid submitted by bidder j in the most recent round and let $\Lambda^n = \left\{l = 1 \ldots L :\ q_l \text{ in } \mathbf{q}_n^h > 0\right\}$, i.e. the set of lot categories for which the bidder has specified positive demand in its most recent headline bid.

In order to identify the lot categories for which prices need to increase, for each omitted bidder $n \in \theta$ we run through the following steps:

(1) For each lot category $l \in \Lambda^n$ we do the following:

- We construct a hypothetical set of bids $\tilde{B}$ that contains all the bids received in the auction phase except for β_n^h (bidder n's headline bid), which we replace with a hypothetical headline bid $\beta_j^{\tilde{h}} = (P_j^{\tilde{h}}; \mathbf{q}_j^{\tilde{h}})$ where q_l is the same as in the original headline bid, and the quantity of all other lots is zero and there the bid amount is adjusted correspondingly, i.e. $P_j^{\tilde{h}} = q_l \cdot p_l$. Thus $\tilde{B} = \left\{ \beta_j^{\tilde{h}} \cup B \setminus \beta_n^h \right\}$.
- We then identify the value-maximising feasible combinations based on the set of bids $\tilde{B}$ and check if bidder n would still be omitted. If this is the cse, we identify l as a lot category that requires a price increment as there would be a clash even if the bidder expressed demand only for lots in this category.

(2) If after running step 1 for bidder n for all lot categories we have not identified any lot category that requires a price increment, then all the lot categories in Λ^n will be identified as requiring a price increment.

We can terminate this process as soon as we found that all lot categories require a price increment.